The limit

$$\lim_{n\to\infty}\left[(n+1)\Big(1+\frac{1}{n}\Big)^{n}-n\Big(1+\frac{1}{n-1}\Big)^{n-1}\right]$$

has the form

$$\lim_{n\to\infty}\big[A(n)B(n)-C(n)D(n)\big],$$

where $B(n)=(1+\frac{1}{n})^{n}$ and $D(n)=(1+\frac{1}{n-1})^{n-1}$ both approach e, but $A(n)=n+1$ and $C(n)=n$ both approach ∞. So the limit has the indeterminate form $\infty-\infty$ and we need to be careful. It's true that we can rewrite the limit as

$$\begin{aligned}&\lim_{n\to\infty}\big[(n+1)B(n)-nD(n)\big]\\ =&\lim_{n\to\infty}\big[n\big(B(n)-D(n)\big)+B(n)\big]\end{aligned}$$

and it's true that $B(n)-D(n)$ approaches 0, but that does not immediately imply that $n\big(B(n)-D(n)\big)$ approaches 0. We need to know something about the *rate* at which $B(n)-D(n)$ approaches 0 (roughly speaking, we need to know that $B(n)-D(n)$ approaches 0 more quickly than $1/n$ does).

The final answer of e is correct, and there are various ways to make a more rigorous argument, but I think it's a more subtle problem than it appears at first glance.

The argument I came up with begins as follows. The original limit is

$$\lim_{n\to\infty}\big[G(n)-G(n-1)\big]$$

where $G(n)=(n+1)^{n+1}/n^{n}$. We can use the mean value theorem to say that

$$G(n)-G(n-1)=\frac{G(n)-G(n-1)}{n-(n-1)}=G'(\xi)$$

for some ξ between $n-1$ and n. We can then use logarithmic differentiation to get an expression for $G'(x)$ that isn't too horrible, and we can use known estimates of logarithms along the way. I can show more details later if there's interest.